Section 17.1

Vector Fields

Definition: Vector Fields in Two Dimensions
Let f and g be defined on a region R of $\mathbb{R}^2$. A **vector field** in $\mathbb{R}^2$ is a function F that assigns to each point in R the vector $\langle f(x,y), g(x,y)\rangle$. That is, F is a multivariable, vector valued function $F : \mathbb{R}^2 \to \mathbb{R}^2$. The vector field is written as

$$\vec{F}(x,y) = \langle f(x,y), g(x,y)\rangle \quad \text{or}$$

$$\vec{F}(x,y) = f(x,y)\vec{\imath} + g(x,y)\vec{\jmath}.$$

A vector field is continuous (differentiable) on R is f and g are continuous (differentiable) on R.

Drawing Flows. A flow is the assignment to each point P in R the vector of $\vec{F}(P)$ with tail at P.

Example 17.1.1. Draw the flow of $\vec{F}(x,y) = \langle 1 - y^2, 0\rangle$ for $|y| \leq 1$.

Radial Vector Fields
Let $\vec{r} = \langle x, y \rangle$ and $f : \mathbb{R}^2 \to \mathbb{R}$. The vector field $\vec{F} = f(x, y)\vec{r}$ is a **radial vector field.**

Example 17.1.2. Draw the flow of the radial vector field $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^3}$

Definition
Let φ be a scalar valued function differentiable on a region of $\mathbb{R}^2$ or $\mathbb{R}^3$. The vector field $\vec{F} = \nabla\varphi$ is a **gradient field** and the function φ is a **potential function** of $\vec{F}$.

The level curves of φ are called the **equipotential curves**.

Example 17.1.3. Find the equipotential curve of $\varphi = \dfrac{x^2 - y^2}{2}$ through $(2, 1)$. Verify that the gradient field of φ is orthogonal to the equipotential curve here.

Definition: Scalar Line Integral
Suppose the scalar-valued function f is defined on a region containing the smooth curve C given by $\vec{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$. The **line integral of f over C** is

$$\int_C f(x(t), y(t))ds = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*))\Delta s_k$$

provided this limit exists over all partitions of $[a, b]$. When this limit exists, f is said to be **integrable** on C.

The differential ds is not a typo! So how to compute??

Example 17.1.4. Find the average temperature of the edge of the plate $R = \{(x, y) : x^2 + y^2 \leq 1\}$ where the temperature is given by $T(x, y) = 100(x^2 + 2y^2)$.

Example 17.1.5. Let C be the path in $\mathbb{R}^3$ given by the union of the line segment from $O(0, 0, 0)$ to $P(1, 0, 0)$ and the line segment from $P(1, 0, 0)$ to $Q(0, 1, 1)$. Evaluate

$$\int_C (xy + 2z)ds.$$